Student seminar solutions Week 14

1. We will start with point *iv.* as we will use it to prove *iii.*.

- *iv.* Let $\mathcal{H} = \Phi(E) = F^\times N_{E/F} J_E$ and $K \supseteq E$. First we show $\rho_{K/F}(\mathcal{H})$ fixes E.
  By Artin Reciprocity, we have that $\mathcal{H} = \ker\rho_{E/F}$. Let $a \in \mathcal{H}$, we have:

  $$\rho_{K/F}(a)|_E = \left(\frac{a}{K/F}\right)|_E \overset{*}{=} \left(\frac{a}{E/F}\right) = \rho_{E/F}(a) = 1.$$

  Where $*$ comes from the consistency property for idèles.
  Now let $x \in K$ fixed by $\rho_{K/F}(\mathcal{H})$, let's show that $x \in E$. Since E is fixed by $\rho_{K/F}(\mathcal{H})$ then so is $E(x)$. Then as before by the consistency property we have for any $a \in \mathcal{H}$

  $$\left(\frac{a}{K/F}\right)|_{E(x)} = \left(\frac{a}{E(x)/F}\right) = 1.$$

  Thus

  $$\Phi(E) = \mathcal{H} = F^\times N_{E/F} J_E \subseteq \ker\rho_{E(x)/F} = F^\times N_{E(x)/F} J_{E(x)} = \Phi(E(x))$$

  Since we have shown $\Phi(E) \subseteq \Phi(E(x))$, then by the ordering theorem we have that $E(x) \subseteq E$ hence $x \in E$.
- *iii.* We will proceed by double inclusion:
  (a) $\supseteq$: This just follows from the ordering theorem, since $K \cap K' \subseteq K$ and $K \cap K' \subseteq K'$ and thus it means that $\Phi(K) \subseteq \Phi(K \cap K')$, $\Phi(K') \subseteq \Phi(K \cap K')$.
  (b) $\subseteq$: By *iv.* we have that K is the fixed field of $\rho_{KK'/F}(\Phi(K))$ and K' is the fixed field of $\rho_{KK'/F}(\Phi(K'))$. Let E be the fixed field of $\rho_{KK'/F}(\Phi(K)\Phi(K'))$. Now since $\Phi(K) \subseteq \Phi(K)\Phi(K')$, we have $\rho_{KK'/F}(\Phi(K)) \subseteq \rho_{KK'/F}(\Phi(K)\Phi(K'))$, so we must have $E \subseteq K$ and as for K', we get that $E \subseteq K \cap K'$.
  Now if $a \in \Phi(K)$ and $a' \in \Phi(K)$, we get :

  $$\left(\frac{aa'}{KK'/F}\right)|_{K\cap K'} = \left(\frac{a}{KK'/F}\right)|_{K\cap K'} \left(\frac{a'}{KK'/F}\right)|_{K\cap K'} = 1.$$

Which means that $\rho_{KK'/F}(\Phi(K)\Phi(K'))$ fixes $K \cap K'$ thus

$$E = K \cap K'.$$

But by *iv*. we know that E is the fixed field of $\rho_{KK'/F}(\Phi(E))$ and by injectivity of the Galois correspondance we have that

$$\rho_{KK'/F}(\Phi(E)) = \rho_{KK'/F}(\Phi(K)\Phi(K')).$$

Now if $a \in \Phi(K \cap K') = \Phi(E)$, $\exists b \in \Phi(K)$ and $b' \in \Phi(K')$ such that $\rho_{KK'/F}(a) = \rho_{KK'/F}(bb')$. Then $\rho_{KK'/F}(a(bb')^{-1}) = 1$ thus $a(bb')^{-1} \in \ker\rho_{KK'/F} = \Phi(KK') = \Phi(K)\cap\Phi(K')$ using point *ii*. of the ordering theorem. So $a(bb')^{-1} = x$ with $x \in \Phi(K)\cap\Phi(K')$ thus

$$a = \underbrace{xb}_{\in\Phi(K)} \underbrace{b'}_{\in\Phi(K')} \in \Phi(K)\Phi(K').$$

Hence the claim follows.

2. Recall that

$$C_F = {}^{J_F}\!\big/_{F^\times} \text{ and } C_{F,S} = {}^{J_{F,S}}\!\big/_{F_S},$$

Since $J_{F,S} \subset J_F$ and $F_S \subset F^\times$, we have a map

$$\iota : C_{F,S} = {}^{J_{F,S}}\!\big/_{F_S} \longrightarrow {}^{J_F}\!\big/_{F^\times} = C_F.$$

This is clear that ι is injective an thus we can see $C_{F,S}$ as a subgroup of C_F.

Let π be the quotient map

$$\pi : J_F \longrightarrow C_F.$$

We want to show that $\pi(J_{F,S}) \cong C_{F,S}$: It's easy to see that $\ker(\pi|_{J_{F,S}}) = J_{F,S} \cap F^\times = F_S$, so by simply using the first isomorphism theorem we get

$$\begin{array}{ccc} J_{F,S} & \longrightarrow & \pi(J_{F,S}) \\ \downarrow & \overset{\cong}{\nearrow} & \\ {}^{J_{F,S}}\!\big/_{F_S} & & \end{array}$$

And thus as we claim:

$$\pi(J_{F,S}) \cong {}^{J_{F,S}}\!\big/_{F_S} = C_{F,S}.$$

Now we define

$$\varphi : \ J_F \xrightarrow{\pi} C_F \xrightarrow{q} {}^{C_F}\!\big/_{C_{F,S}}.$$

The map is clearly surjective and, we will show by double inclusion that its kernel is $F^\times J_{F,S}$:
First, let $x \in \ker(\varphi)$, then $\pi(x) \in C_{F,S} \cong \pi(J_{F,S})$. So $\exists\, y \in J_{F,S}$ such that $\pi(x) = \pi(y)$ thus $xy^{-1} \in F^\times$ and $\exists z \in F^\times$ such that $x = zy \in F^\times J_{F,S}$.
Now, let $x \in F^\times J_{F,S}$, then $\exists z \in F^\times, y \in J_{F,S}$ such that $x = zy$, thus $\varphi(x) = \varphi(zy) = q \circ \pi(zy) = q(\pi(y)) = 1$. Thus $\ker\varphi = F^\times J_{F,S}$ and applying the first isomorphism theorem:

$$J_F/F^\times J_{F,S} \cong C_F/C_{F,S}.$$

This gives us the algebraic isomorphism. For the topological part, we have to show that φ is a homeomorphism. All maps involved are quotient, thus continuous. Now since J_F is a locally compact topological group and $F^\times$ is discrete, π is also an open map. Then since $J_{f,S}$ is an open subgroup and π is an open map, we have that $\pi(J_{F,S}) = C_{F,S}$ is also open, hence closed in C_F hence the quotient map q is also open. As φ is the composition of continuous open maps, φ is also a continuous open map which is bijective, thus this is a homeomorphism.

3. (a) We show that there is an exact sequence

$$1 \longrightarrow \mathcal{O}_F^\times \longrightarrow F_S \xrightarrow{\gamma} I_{S_0}.$$

We check that this is well defined, let $\alpha \in F_S$, by definition of F_S, $v(\alpha) = 0, \forall v \notin S_0$ hence $< \alpha >\in I_{S_0}$.
To show exactness we have to show that $\ker\gamma = \mathcal{O}_F^\times$ (since the injectivity of the first map is obvious) :
If $\alpha \in \mathcal{O}_F^\times$, then $< \alpha >= \mathcal{O}_F$, so $\gamma(\alpha) = 1$. Hence $\mathcal{O}_F^\times \subset \ker\gamma$.
Now if $\alpha \in \ker\gamma$, then $< \alpha >= \mathcal{O}_F$, thus $\alpha \in \mathcal{O}_F^\times$ so $\ker\gamma = \mathcal{O}_F^\times$. This proves exactness.

(b) Let $h = \#\mathcal{C}_F$. For$v \in S_0$, the ideal $\mathfrak{p}_v^h$ is clearly principal, so this means that $\exists \alpha_v \in F^\times$ such that

$$< \alpha_v >= \mathfrak{p}_v^h.$$

Then it's easy to see that $\alpha_v \in F_S$, thus $\mathfrak{p}_v^h \in \gamma(F_S)$. So:

$$I_{S_0}^h \subset \gamma(F_S) \subset I_{S_0}.$$

Since both I_{S_0} and $I_{S_0}^h$ are free Abelian of rank $\#S_0$ so is $\gamma(F_S)$.

(c) From part a) we have the s.e.s

$$1 \longrightarrow \mathcal{O}_F^\times \longrightarrow F_S \xrightarrow{\gamma} \gamma(F_S) \longrightarrow 1.$$

from part b), the group $\gamma(F_S)$ is free Abelian. So this tells us that the sequence splits, and it follows that

$$F_S \cong \mathcal{O}_F^\times \times \gamma(F_S).$$

(d) Dirichlet's Unit Theorem tells us that

$$\mathcal{O}_F^\times \cong W_F \times \mathbb{Z}^{r_1+r_2-1},$$

And from part c):

$$F_S \cong W_F \times \mathbb{Z}^{r_1+r_2-1} \times \gamma(F_S).$$

From part b) $\gamma(F_S) \cong \mathbb{Z}^{\#S_0}$

$$F_S \cong W_F \times \mathbb{Z}^{r_1+r_2-1+\#S_0}.$$

as $\#S_\infty = r_1 + r_2$ and $\#S_0 = \#S - \#S_\infty$ we compute and conclude that:

$$F_S \cong W_F \times \mathbb{Z}^{\#S-1}.$$

4. • a) Since $\mu_2 = \{\pm 1\} \subset \mathbb{Q}$, we may apply Kummer theory for $n = 2$ over $\mathbb{Q}$. Let $a \in \mathbb{Q}^\times$ and look at the field $K = \mathbb{Q}(\sqrt{a})$. If $a \in (\mathbb{Q}^\times)^2$, then $\sqrt{a} \in \mathbb{Q}$ and $K = \mathbb{Q}$. Otherwise, $K/\mathbb{Q}$ is a quadratic extension, so it is cyclic of degree 2 and $\mathrm{Gal}(K/\mathbb{Q})$ has exponent 2.

   Conversely, any cyclic extension of $\mathbb{Q}$ of degree 2 is of the form $K = \mathbb{Q}(\sqrt{a})$ for some $a \in \mathbb{Q}^\times$, with a determined up to squares. This shows that quadratic extensions of $\mathbb{Q}$ correspond to nontrivial elements of $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$.
   Now, let $a_1, \ldots, a_r \in \mathbb{Q}^\times - (\mathbb{Q}^\times)^2$ and consider

$$K = \mathbb{Q}(\sqrt{a_1}, \ldots, \sqrt{a_r}).$$

   This extension is Abelian. An automorphism is determined by choosing the sign of each $\sqrt{a_i}$, so the Galois group has exponent 2. If the classes of the a_i are linearly independent modulo squares, then

$$(Gal)(K/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^r.$$

   Thus the Kummer 2-extensions of $\mathbb{Q}$ are exactly of the form $K = \mathbb{Q}(\sqrt{a_1}, \ldots, \sqrt{a_r})$ whith $a_1, \ldots, a_r \in \mathbb{Q}^\times - (\mathbb{Q}^\times)^2$ linearly independent.

• b)Let $F = \mathbb{Q}(\zeta_3)$. Since $\mu_3 \subset F$, we can apply Kummer theory for $n = 3$. Let $\alpha \in F^\times$ and $K = F(\sqrt[3]{\alpha})$. If $\alpha \in (F^\times)^3$, then $\sqrt[3]{\alpha} \in F$ and $K = F$. Otherwise, K/F is a cyclic extension of degree 3, and $\mathrm{Gal}l(K/F)$ has exponent 3. Thus using the same argumentation as for a) , the cyclic extensions of degree 3 over F are classified by the nontrivial classes in $F^\times/(F^\times)^3$.
   Now let $\alpha_1, \ldots, \alpha_r \in F^\times - (F^\times)^3$ and consider

$$K = F(\sqrt[3]{\alpha_1}, \ldots, \sqrt[3]{\alpha_r}).$$

The extension K/F is Abelian, and each automorphism sends $\sqrt[3]{\alpha_i} \mapsto \zeta_3^k \sqrt[3]{\alpha_i}$ with $k = 1, 2, 3$. So the Galois group has exponent 3and if the classes of the $\alpha_i's$ are linearly independent, we get

$$\mathrm{Gal}(K/F) \cong (\mathbb{Z}/3\mathbb{Z})^r.$$

So the Kummer 3-extensions of $\mathbb{Q}(\zeta_3)$ are of the form $K = F(\sqrt[3]{\alpha_1}, \dots, \sqrt[3]{\alpha_r})$ with $\alpha_1, \dots, \alpha_r \in F^\times - (F^\times)^3$.

5. We will do a double inclusion:

   (a) $\supseteq$ This is the easy inclusion: Pick $x \in F_S^n$, then $\exists y \in F_S$ such that $x = y^n$. Since $y \in F_S$, we have $y \in U_v$ for all $v \notin S$, hence $x = y^n \in U_v$. For $v \in S$, $x \in (F_v^\times)^n$. Thus $x \in B$ by definition of B, and since $x \in F^\times$, $x \in B \cap F^\times$.

   (b) $\subseteq$ Let $x \in B \cap F^\times$. Then $x \in (F_v^\times)^n$ for all $v \in S$, so this means that:

   $$[F_v(x^{1/n}) : F_v] = 1 \text{ for all } v \in S.$$

   this shows that $\mathfrak{p}_v$ splits completely in $F(x^{1/n})/F$ for all $v \in S$.
   If $v \notin S$, we get that $x \in U_v$, moreover since $v \nmid n$ and $\mu_n \subset F$,then $\mathfrak{p}_v$ is unramified in $F(x^{1/n})$. Recall that

   $$J_{F,S} = \prod_{v \in S} F_v^\times \times \prod_{v \notin S} U_v$$

   For $v \in S$ pick w a place above v then the local norm map $N_{F(x^{1/n})_w/F_v}$ is the identity and thus $F_v^\times \subset N_{F(x^{1/n})/F}(J_{F(x^{1/n})})$ and for $v \notin S$ and w a place above v, since S contains all infinite places, then the local extension $F(x^{1/n})_w/F_v$ is a finite extension. Then we know from exercises 4, sheet 10 that the norm map is surjective for finite extension of fields, thus the local norm map is surjective and we get $N_{F(x^{1/n})_w/F_v}(U_w) = U_v$ hence every units in F_v is the image of a norm of $F_w^\times$. Thus $U_v \subset N_{F(x^{1/n})/F}(J_{F(x^{1/n})})$ and so we have that :

   $$J_{F,S} \subset N_{F(x^{1/n})/F}(J_{F(x^{1/n})}).$$

   And since $J_F = F^\times J_{F,S}$, we get

   $$J_F = F^\times J_{F,S} \subset F^\times N_{F(x^{1/n})/F}(J_{F(x^{1/n})}).$$

   By Artin reciprocity, the kernel of the Artin map

   $$\rho_{F(x^{1/n})/F} : J_F \longrightarrow \mathrm{Gal}\left(F(x^{1/n})/F\right)$$

   is equal to $F^\times N_{F(x^{1/n})/F}(J_{F(x^{1/n})})$. So this means that $\ker\rho_{F(x^{1/n})/F} = J_F$, and thus

   $$\mathrm{Gal}\left(F(x^{1/n})/F\right) \cong {}^{J_F}\!\diagup_{N_{F(x^{1/n})/F}(J_{F(x^{1/n})})} = {}^{J_F}\!\diagup_{J_F} = 1.$$

Hence $F(x^{1/n}) = F$ and $x^{1/n} \in F$.
Since $x \in B$, we also have $x \in U_v$ for all $v \notin S$, and therefore $x \in F_S$ thus $x \in F_S^n$.

This concludes the proof.